Section: Eigenvalues and Eigenvectors

Applications (For next 8-9 lectures)

- Dynamical systems: predator/prey, RLC & SPRING/MASS/DAMPER
- Control systems
- Markov processes, population dynamics, markov chains and baseball statistics
- Opinion dynamics in social media

Topics:

- Eigenvalues and Eigenvectors (ALA 8.2)
  - Teaser: Repeated and Complex Eigenvalues
  - Basic Properties
- Eigenvector Bases (ALA 8.3)
  - Similar matrices
  - Diagonalization (over the reals)
  - Applications to dynamical systems (ALA 10.1)

Additional reading: LAA 5.1 and 5.2.

Repeated and Complex Eigenvalues

Recall that at the end of last class, we saw an example of a 3×3 matrix with a double eigenvalue:

$$A = \left[\begin{array}{c|cc} 2 & -1 & -1 \\ \hline 0 & 3 & 1 \\ 0 & 1 & 3 \end{array}\right], \quad \text{with} \quad \lambda_1 = 2, \quad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{\hat{v}}_1 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$\lambda_2 = 4, \quad \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

In this case, even though A only has 2 distinct eigenvalues, it still has three linearly independent eigenvectors: as we'll see later, this is important as it will allow us to use the eigenvectors of A to define a basis for $\mathbb{R}^3$.

This doesn't always happen though. Next, we'll see a simple example of a 2×2 matrix with only one eigenvector!

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. Then A has a double eigenvalue at $\lambda_1 = 2$,

because $\det(A - \lambda I) = (2-\lambda)^2 = 0 \iff \lambda = 2$.

The associated eigenvector equation $(A - 2I)\underline{v} = \underline{0}$ then becomes:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = 0, \; v_1 \text{ free}$$

i.e., $\underline{v} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector, and we set $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus, even though $\lambda = 2$ is a double eigenvalue, it only admits a one dimensional eigenspace. The list of eigenvalues/vectors is in a sense incomplete.

For the next few lectures, we will avoid such degenerate examples, but we will need how to handle them when we return to our motivating application of linear dynamical systems.

So far, all of the examples we've considered have had <u>real</u> eigenvalues. In general, however, complex eigenvalues (and eigenvectors) are also important.

ONLINE NOTES: Provide link to review of complex numbers.

Example Consider the 2x2 matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ corresponding to a $90°$ rotation.

Its eigenvalues are determined by solutions to

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0,$$

i.e., $\lambda^2 = -1$. There is no $\lambda \in \mathbb{R}$ satisfying this equation, but as you know, this means we have to expand our candidate solutions to include complex numbers. In this case, $\lambda_1 = +i$ and $\lambda_2 = -i$, for $i = \sqrt{-1}$ the imaginary number, are two solutions.

The corresponding eigenvector equation, which now will include complex numbers, becomes:

For $\lambda_1 = i$: $(A - iI)\underline{v} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -iv_1 - v_2 = 0 \\ v_1 - iv_2 = 0 \end{matrix} \Rightarrow v_2 = -iv_1$

so that $\underline{v} = a \begin{bmatrix} 1 \\ -i \end{bmatrix}$ for any $a \in \mathbb{C}$, and we set $\underline{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ as the eigenvector associated with $\lambda_1 = i$.

For $\lambda_2 = -i$: following the same procedure, we see that $\underline{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is the corresponding eigenvector.

Thus, we have

$$\lambda_1 = i, \ \underline{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{and} \quad \lambda_2 = -i, \ \underline{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

A couple of observations:

- λ_1 and λ_2 are complex conjugates, i.e., $\lambda_1 = \bar{\lambda}_2$, as are $\underline{v}_1$ and $\underline{v}_2$. This is a general fact about real matrices:

Theorem: If A is a real matrix (i.e., $A_{ij} \in \mathbb{R}$), with a complex eigenvalue $\lambda = a + ib$ and corresponding complex eigenvector $\underline{v} = \underline{x} + i\underline{y}$, then the complex conjugate $\bar{\lambda} = a - ib$ is also an eigenvalue with complex conjugate eigenvector $\bar{\underline{v}} = \underline{x} - i\underline{y}$.

- The eigenvalues for our example, which defines a pure rotation in $\mathbb{R}^2$, are purely imaginary. This isn't a coincidence! When we discuss symmetric and skew symmetric matrices later in this class, this will be further explained, but for now, you should start associating imaginary components of eigenvalues with rotations.

Basic Properties of Eigenvalues

We won't belabor the derivation of the following properties of eigenvalues: they mostly follow from properties of the determinant and the Fundamental Theorem of Algebra. The TLDR is that for any $A \in \mathbb{R}^{n \times n}$, its characteristic polynomial can be factored as:

$$\det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

where the complex numbers $\lambda_1, \ldots, \lambda_n$, some of which may be repeated, are the eigenvalues of A. Therefore, we immediately conclude that:

Theorem: An $n \times n$ real matrix has at least one, and at most n, distinct complex eigenvalues.

Another useful property is that a matrix A and its transpose A^T have the same eigenvalues. This follows from another property of the determinant:

Fact 5: $\det A = \det A^T$.

This means that both A and A^T have the same characteristic polynomial & hence eigenvalues. They do not however have the same eigenvectors!

Eigenvector Bases

Most of the vector space bases that are useful in applications are assembled from the eigenvectors of a particular matrix. In this section, we focus on matrices with a "complete" set of eigenvectors, and show how these form a basis for $\mathbb{R}^n$ (or in the complex case, $\mathbb{C}^n$). Such eigenbases allow us to rewrite the linear transformation determined by a matrix in a simple diagonal form — matrices that allow us to do this are called diagonalizable. We focus on matrices with real eigenvalues and eigenvectors to start, and will return to matrices with complex eigenvalues/vectors next class.

Our starting point is the following theorem, which we will state as a fact. It is a generalization of the pattern we saw above that distinct eigenvalues have linearly independent eigenvectors.

Theorem: If the matrix $A \in \mathbb{R}^{n\times n}$ has n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$, then the corresponding real eigenvectors $\underline{v}_1, \dots, \underline{v}_n$ form a basis for $\mathbb{R}^n$.

Example: Recall from last lecture that we saw that $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalue/vector pairs

$$\lambda_1 = 4, \ \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 2, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$v_1, v_2 \in \mathbb{R}^2$ are linearly independent, and hence form a basis for $\mathbb{R}^2$ since $\dim \mathbb{R}^2 = 2$.

However, we also saw an example where a 3x3 matrix only had two distinct eigenvalues, but still had three linearly independent eigenvectors:

Example: Recall the 3x3 matrix $A = \left[\begin{array}{c|cc} 2 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{array}\right]$. We showed it had the following eigenvalue/vector pairs:

$$\lambda_1 = 2, \quad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\underline{v}}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{and} \quad \lambda_2 = 4, \quad \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

The collection $\underline{v}_1, \hat{\underline{v}}_1, \underline{v}_2 \in \mathbb{R}^3$ are linearly independent, and hence form a basis for $\mathbb{R}^3$ since $\dim \mathbb{R}^3 = 3$.

Notice that in this last example $\dim V_{\lambda_1} = 2$ (why?) for the double eigenvalue $\lambda_1 = 2$, and similarly, $\dim V_{\lambda_2} = 1$ for the simple eigenvalue $\lambda_2 = 4$, so that there is a "new" eigenvector for each time an eigenvalue appears as a factor of the characteristic polynomial.

In general, the number of times an eigenvalue λ_i appears as a solution to the characteristic polynomial is called its algebraic multiplicity, whereas the dimension of its eigenspace $\dim V_\lambda$ is called its geometric multiplicity. Our observation is that if these two numbers match for each eigenvalue, then we can form a basis for $\mathbb{R}^n$.

Theorem: The eigenvectors of a matrix $A \in \mathbb{R}^{n\times n}$ form a basis for $\mathbb{R}^n$ if and only if, for each distinct eigenvalue λ_i, the geometric multiplicity of λ_i matches its geometric multiplicity $\dim V_{\lambda_i}$.

For the next little bit, we will assume that our matrix A satisfies the above theorem. What does this buy us? To answer this question, we need to introduce the idea of similar transformations.

Given a vector $\underline{x} \in \mathbb{R}^n$ with coordinates x_i with respect to the standard basis, i.e., $\underline{x} = x_1\underline{e_1} + x_2\underline{e_2} + \cdots + x_n\underline{e_n}$, we can find the coordinates $y_1, \ldots, y_n$ of $\underline{x}$ with respect to a new basis $\underline{b_1}, \ldots, \underline{b_n}$ by solving the following linear system:

$$y_1\underline{b_1} + y_2\underline{b_2} + \cdots + y_n\underline{b_n} = \underline{x} \iff B\underline{y} = \underline{x},$$

where $B = [\underline{b_1}\ \underline{b_2} \cdots \underline{b_n}]$. Since the $\underline{b_i}$ form a basis of $\mathbb{R}^n$, they are linearly independent, which means that B is nonsingular.

Now, suppose I have a matrix $A \in \mathbb{R}^{n\times n}$, which I use to define the linear transformation $f: \mathbb{R}^n \to \mathbb{R}^n$ given by $f(\underline{x}) = A\underline{x}$. Here f's inputs $\underline{x} \in \mathbb{R}^n$ and outputs $f(\underline{x}) \in \mathbb{R}^n$ are both expressed with respect to the standard basis $\underline{e_1}, \ldots, \underline{e_n}$, and its matrix representative is A.

What if we would like to implement this linear transformation with respect to the basis B, that is, define a function $g: \mathbb{R}^n \to \mathbb{R}^n$ with inputs $\underline{y} \in \mathbb{R}^n$ in B-coordinates, and outputs $g(\underline{y}) \in \mathbb{R}^n$ in B-coordinates. To accomplish this, we need to convert both the input $\underline{x}$ and output $f(\underline{x})$ into B-coordinates.

Relating inputs $\underline{x}$ to B-coordinate inputs $\underline{y}$ is easy: $\underline{x} = B\underline{y}$

Relating outputs $f(\underline{x})$ to B-coordinate outputs $g(\underline{y})$ is too: $f(\underline{x}) = Bg(\underline{y})$

Putting these together, we see that

$$f(\underline{x}) = A\underline{x} \iff Bg(\underline{y}) = AB\underline{y}$$

which lets us solve for $g(\underline{y}) = B^{-1}AB\underline{y}$.

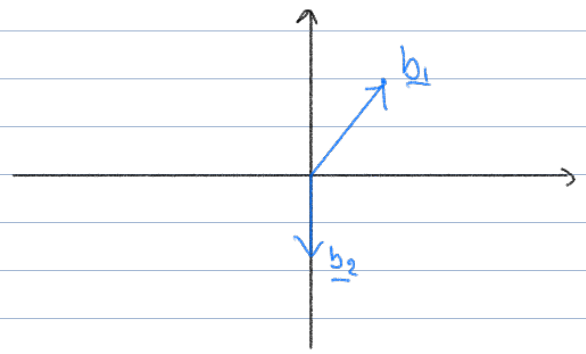
We conclude that if A is the matrix representation of a linear transformation in the standard basis, then $B^{-1}AB$ is the matrix representation in the basis B.

(B-coordinates) (standard coordinates) (B-coordinates)

$$\underline{y} \xrightarrow{B} \underline{x} = B\underline{y} \xrightarrow{A} A\underline{x} = AB\underline{y} \xrightarrow{B^{-1}} B^{-1}AB\underline{y}$$

<u>Example</u>: Consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $f(\underline{x}) = A\underline{x}$. This transformation maps $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$. Consider the basis $\underline{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\underline{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

illustrated in blue below:



The basis matrix B is $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, and $B^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. The matrix representation for $g(\underline{y})$ is then

$$B^{-1}AB = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix},$$

and the map $g(\underline{y}) = B^{-1}AB\underline{y}$ takes $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} 5y_1 + 2y_2 \\ -8y_1 - 3y_2 \end{bmatrix}$.

In the above example, our change of basis didn't really help us understand what the linear transformation $f(\underline{x})$ is doing any better than our starting point. However, we'll see now that if we use the basis defined by the eigenvectors of a matrix, some magic happens! We'll start with an example, and then extract out a general conclusion.

<u>Example</u>: Consider the linear transformation $h(x_1, x_2) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$. It has matrix representation $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ with respect to the standard basis of $\mathbb{R}^2$.

The eigenvalues of A are computed by solving $\det(A - \lambda I) = 0$:

$$\det\begin{bmatrix}1-\lambda & -1\\ 2 & 4-\lambda\end{bmatrix} = (1-\lambda)(4-\lambda)+2 = \lambda^2 - 5\lambda + 6$$
$$= (\lambda-2)(\lambda-3) = 0$$

so that $\lambda_1 = 2$ and $\lambda_2 = 3$. Solving the appropriate eigenvector equations $(A-\lambda_i I)\underline{v}_i = \underline{0}$, we obtain the following eigenvalue/eigenvector pairs:

$$\lambda_1 = 2,\ \underline{v}_1 = \begin{bmatrix}1\\-1\end{bmatrix} \quad \text{and} \quad \lambda_2 = 3,\ \underline{v}_2 = \begin{bmatrix}1\\-2\end{bmatrix}.$$

Let's see what happens if we express A in the coordinate system defined by the eigenbasis $V = [\underline{v}_1\ \underline{v}_2] = \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}$.

First, we compute $V^{-1} = \dfrac{1}{1\cdot(-2)-(1)(-1)}\begin{bmatrix}-2 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}$, and then

find $V^{-1}AV = \begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}.$

This matrix is diagonal! This means it applies a simple stretching action in the coordinates defined by the eigenvectors. The eigenvalues for this new matrix are also $\lambda_1 = 2$ and $\lambda_2 = 3$, but in this case, eigenvectors are much simpler: $\hat{\underline{v}}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\hat{\underline{v}}_2 = \begin{bmatrix}0\\1\end{bmatrix}$.

This example showed us an example of a very important property of an eigenbasis: they diagonalize the original matrix representative! Working with diagonal matrices is very convenient, and thus diagonalization is very useful when we can do it.

Although we only saw a 2x2 example, the idea is applicable to general $n\times n$ matrices. We say that a square matrix A is diagonalizable if there exists a nonsingular matrix V and a diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ such that

$$V^{-1}AV = \Lambda \quad \text{or equivalently} \quad A = V\Lambda V^{-1} \qquad (D)$$

Let's try to understand condition (D) a little bit more by writing it as

$$AV = V\Lambda.$$

Now, for $V = [\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_n]$, this becomes: $[A\underline{v}_1\ A\underline{v}_2 \cdots A\underline{v}_n] = [\lambda_1\underline{v}_1\ \lambda_2\underline{v}_2 \cdots \lambda_n\underline{v}_n]$;

Focusing on the k^{th} column of this $n \times n$ matrix equation, we see something familiar:

$$A \underline{v}_k = \lambda_k \underline{v}_k,$$

that is, the columns of V must be eigenvectors, and the diagonal elements λ_i must be eigenvectors!

Therefore, we immediately get the following characterization of when a matrix is diagonalizable:

Theorem: A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if it has n linearly independent eigenvalues.

ONLINE NOTES: worked examples of matrices that are diagonalizable and not.

Application to Linear ODEs

Let's return to our motivating application of linear time invariant homogeneous first order dynamical systems:

$$\frac{d}{dt}\underline{u} = A\underline{u}, \qquad \text{(LTI)}$$

where here the solution $\underline{u}(t)$ parameterizes a curve in $\mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$ is a known constant matrix.

We can reduce solving (LTI) to solving n-independent scalar ODEs when the matrix A is diagonalizable. Let $V = [\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_n]$ be the $n \times n$ matrix of the eigenvectors of A, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ the diagonal matrix of corresponding eigenvalues. Then $A = V \Lambda V^{-1}$ so that

$$\frac{d}{dt}\underline{u} = V \Lambda V^{-1} \underline{u}. \quad \text{(S1)}$$

As above, we set $V\underline{y} = \underline{u}$ so that $\underline{y} = V^{-1}\underline{u}$ and $\frac{d\underline{u}}{dt} = V\frac{d\underline{y}}{dt}$ to rewrite (S1) as

$$V \frac{d\underline{y}}{dt} = V \Lambda \underline{y}. \quad \text{(S2)}$$

Since V is nonsingular, (S2) is true if and only if

$$\frac{d\underline{y}}{dt} = \Lambda \underline{y} \quad \text{(S3)}.$$

Thus by working in the coordinate system defined by the eigenbasis V, we've reduced solving (LTI) to solving the n decoupled scalar odes in (S3):

$$\frac{dy_i}{dt} = \lambda_i y_i. \quad (Si)$$

The general solution to (Si) is $y_i(t) = c_i e^{\lambda_i(t-t_0)}$, with $c_i = y_i(t_0)$. Here, $y_i(t_0)$ can be computed via $\underline{y}(t_0) = V^{-1}\underline{u}(t_0)$.

Now, given the solution $\underline{y}(t) = (y_1(t), \ldots, y_n(t))$ in the eigen basis V, we need to map it back to our original coordinates via

$$\underline{u}(t) = V\underline{y}(t) = c_1 e^{\lambda_1(t-t_0)}\underline{v}_1 + c_2 e^{\lambda_2(t-t_0)}\underline{v}_2 + \cdots + c_n e^{\lambda_n(t-t_0)}\underline{v}_n \quad (SOL)$$

where we remember that $\underline{c} = V^{-1}u(t_0)$.

We just showed something incredibly powerful: any solution to (LTI) is a linear combination of the functions

$$\{e^{\lambda_1(t-t_0)}\underline{v}_1, e^{\lambda_2(t-t_0)}\underline{v}_2, \ldots, e^{\lambda_n(t-t_0)}\underline{v}_n\}, \quad (*)$$

i.e., these form a basis for the solutions of (LTI); which particular solution we select is specified by the initial conditions via $\underline{c} = V^{-1}u(t_0)$.

<u>Example</u>: Consider $\frac{d}{dt}\underline{u} = A\underline{u}$, with $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. From our previous example, we know A has eigenvalue/eigenvector pairs:

$$\lambda_1 = 2, \ \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 3, \ \underline{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix};$$

and therefore solutions $\underline{u}(t)$ take the form

$$\underline{u}(t) = c_1 e^{\lambda_1(t-t_0)}\underline{v}_1 + c_2 e^{\lambda_2(t-t_0)}\underline{v}_2 = c_1 e^{2(t-t_0)}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3(t-t_0)}\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Suppose we have the requirement that $\underline{u}(t_0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, then we solve

$$u(t_0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \end{bmatrix}$, and obtain the specific solution $\underline{u}(t) = 10e^{2(t-t_0)}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 7e^{3(t-t_0)}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

A few final comments:

1) (SOL) shows that a solution $\underline{u}(t)$ is a sum of exponential functions "in the direction" of the eigenvectors $\underline{v}_i$: these functions either decay ($\lambda_i < 0$), explode ($\lambda_i > 0$) or stay constant ($\lambda_i = 0$).

2) We have assumed real eigenvalues and eigenvectors throughout. It turns out our analysis holds true even when eigenvalues/vectors are complex; however, interpreting the results as solutions requires a little more care (we'll address this next class).

3) (SOL) does not hold if a matrix is not diagonalizable. We'll see a brief preview of how to deal with matrices we can't diagonalize next class (you'll see much more in ESE 2100).